29 The AdS/CFT Correspondence

29.1 Waves on AdS_5

The conjectured corespondence is

$$\langle \exp \int d^4x \phi_0(x) \mathcal{O}(x) \rangle_{\mathrm{CFT}} = Z_{\mathrm{string}} \left[z^{\Delta-4} \phi(x,z)|_{z=0} = \phi_0(x) \right] \ . \quad (29.1)$$

For large N and $g^2 N$,

$$Z_{\mathrm{string}} \approx e^{-S_{\mathrm{sugra}}} \quad (29.2)$$

Consider a massive scalar on AdS_5:

$$S = \frac{1}{2} \int d^4x dz \sqrt{g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) \quad (29.3)$$

Using the metric

$$ds^2 = \frac{R^2}{z^2} (dz^2 + \sum_{i=1}^{4} dx^i dx^i) \quad (29.4)$$

and assuming a factorized solution of the form

$$\phi(x,z) = e^{ip.x} f(pz) \quad (29.5)$$

we find

$$z^5 \partial_z \left(\frac{1}{z^3} \partial_z f \right) - z^2 p^2 f - m^2 R^2 f = 0 \quad (29.6)$$

Writing $y = pz$ the solutions are

$$f(y) = \left\{ \begin{array}{ccc} y^2 I_{\Delta-2}(y) & \sim & y^{\Delta}, \text{ as } y \to 0 \\ y^2 K_{\Delta-2}(y) & \sim & y^{4-\Delta}, \text{ as } y \to 0 \end{array} \right. \quad (29.7)$$

where

$$\Delta = 2 + \sqrt{4 + m^2 R^2} \quad (29.8)$$

and $I_{\Delta-2}(y)$ and $K_{\Delta-2}(y)$ are modified Bessel functions. $I_{\Delta-2}(y)$ blows up as $y \to 0$ so it does not correspond to a finite action. If we apply a scaling transformation

$$x \to \frac{x}{\rho} \quad (29.9)$$

$$p \to \rho p \quad (29.10)$$

then

$$\phi(x,z) \to \rho^{4-\Delta} e^{ip.x} f(pz) \tag{29.11}$$

so we see this solution has conformal weight $4-\Delta$, so the CFT operator that it couples to on the boundary must have dimension Δ.

Using a δ function source on the boundary rather than a plane wave one finds

$$\phi(x,z) = c \int d^4x' \frac{z^\Delta}{(z^2+|x-x'|^2)^\Delta} \phi_0(x') \tag{29.12}$$

which scales as $z^{4-\Delta}\phi_0(x)$ for small z. We also have for small z:

$$\partial_z \phi(x,z) = c\Delta \int d^4x' \frac{z^{\Delta-1}}{|x-x'|^{2\Delta}} \phi_0(x') + \mathcal{O}(z^{\Delta+1}) \tag{29.13}$$

Integrating the action by parts, and using the equation of motion, yields:

$$\begin{aligned} S &= \frac{1}{2}\int d^4x dz\, \partial_5 \left(\frac{r^3}{z^3}\phi \partial_5 \phi\right) \\ &= \frac{c\Delta R^3}{2}\int d^4x d^4x' \frac{\phi_0(x)\phi_0(x')}{|x-x'|^{2\Delta}} \end{aligned} \tag{29.14}$$

so

$$\begin{aligned} \langle \mathcal{O}(x)\mathcal{O}(x')\rangle &= \frac{\delta^2 S}{\delta\phi_0(x)\,\delta\phi_0(x')} \\ &= \frac{c\Delta R^3}{|x-x'|^{2\Delta}} \end{aligned} \tag{29.15}$$

as expected for an operator of dimension Δ in a conformal theory.

In AdS_{d+1} one always finds dimensions related to masses:

$$\begin{aligned} \text{scalars}: &\quad \Delta_\pm = \tfrac{1}{2}(d \pm \sqrt{d^2+4m^2R^2}) \\ \text{vectors}: &\quad \Delta_\pm = \tfrac{1}{2}(d \pm \sqrt{(d-2)^2+4m^2R^2}) \\ p\,\text{forms}: &\quad \Delta_\pm = \tfrac{1}{2}(d \pm \sqrt{(d-2p)^2+4m^2R^2}) \\ \text{massless spin 2}: &\quad \Delta = d \end{aligned}$$

The relation between mass in AdS_{d+1} and operator dimensions in the boundary CFT is expected to hold for stringy states as well

$$m \sim \frac{1}{l_s} \leftrightarrow \Delta \sim (g^2 N)^{\frac{1}{4}} \tag{29.16}$$

$$m \sim \frac{1}{l_{\mathrm{Pl}}} \leftrightarrow \Delta \sim N^{\frac{1}{4}} \tag{29.17}$$

which for large N and large $g^2 N$ correspond to very large dimension operators which we neglect in the supergravity approximation.

29.2 Spectrum of Operators in the CFT

We are mainly interested in chiral primary operators. Recall that the dimensions of chiral operators can be calculated from their R charge. Primary operators are those which are annihilated by superconformal lowering operators S_α and K_μ, they are the lowest dimension operators in the superconformal multiplet, other operators (descendant operators) can be obtained by acting with superconformal raising operators Q_α and P_μ. A few examples of chiral primary operators are (using $\mathcal{N}=1$ and $\mathcal{N}=0$ language):

- $\mathrm{Tr}(\Phi^{I_1}...\Phi^{I_k})$ when symmetrized traceless in the $SU(4)_R$ indices I_k. These operators transform under $SU(4)_R$ as $(0,k,0)$ (e.g. $\mathbf{20'}$, $\mathbf{50}$, $\ldots$) and have dimension $\Delta = k$.
- $\mathrm{Tr}(W_\alpha W^\alpha \Phi^{I_1}...\Phi^{I_k})$ where W_α is the field strength chiral superfield. They transform under $SU(4)_R$ as $(0,k,2)$ (e.g. $\mathbf{10}$, $\mathbf{45}$, $\ldots$) and have dimension $\Delta = k+3$.
- $\mathrm{Tr}\phi^k F^2 + ...$ They transform under $SU(4)_R$ as $(0,k,0)$ (e.g. $\mathbf{1}$, $\mathbf{1}$, $\mathbf{20'}$, $\ldots$) and have dimension $\Delta = k+4$.
- J^μ_R the R charge current transforms as a $\mathbf{15}$ and has dimension $\Delta = 3$
- $T^{\mu\nu}$ the stress-energy tensor transforms as a $\mathbf{1}$ and has dimension $\Delta = 4$

29.3 Type IIB String theory on $\mathrm{AdS}_5 \times S^5$

The spectrum of Kaluza-Klein harmonics of supergravity on $AdS_5 \times S^5$ is derived by examining the Kaluza-Klein (spherical) harmonics on S^5, which fall into irreducible representations of $SU(4) \sim SO(6)$, with masses determined by their $SU(4)$ quantum numbers. Low mass representations include

- a spin-2 family with $m^2R^2 = k(k+4)$, $k \geq 0$, which tranform under $SU(4)$ as $(0,k,0)$, (e.g. $\mathbf{1}, \mathbf{6}, \mathbf{20'}, ...$). $k=0$ corresponds to the graviton which couples to the stress-energy tensor

- a spin-1 family with $m^2R^2 = (k-1)(k+1)$, $k \geq 1$, which tranform under $SU(4)$ as $(1, k-1, 1)$ (e.g. $\mathbf{15}, \mathbf{64}, \mathbf{175}, ...$). $k=1$ correspond to the gauge bosons of $SU(4)$ and couple to the R current.

- a spin-0 family with $m^2 = k(k-4)$, $k \geq 2$, labeled by $(0,k,0)$ $(\mathbf{20'}, \mathbf{50}, \mathbf{105}, ...)$. They couple to $\mathrm{Tr}(\Phi^{1_1} ... \Phi^{i_k})$

- a complex spin-0 family with $m^2 = (k-1)(k+3)$, $k \geq 0$ labeled by $(0,k,2)$ ($\mathbf{10}, \mathbf{45}, \mathbf{126}, ...$). They couple to $\mathrm{Tr}(W_\alpha W^\alpha \Phi^{1_1} ... \Phi^{i_k})$.

- a complex spin-0 family with $m^2 = k(k+4)$, $k \geq 0$, labeled by $(0,k,0)$ $(\mathbf{1}, \mathbf{6}, \mathbf{20'}, ...)$. They couple to $\mathrm{Tr}\phi^k F^2 + ...$ where a is The massless $(k=0)$ mode is the dilaton which couples to TrF^2.

The graviton, the massless gauge bosons, and the scalars in the above three families in the representations $\mathbf{20'}, \mathbf{10}, \mathbf{1}$ of $SU(4)$ are in the same multiplet.

The lowest states of the these KK families correspond to bosons in the gauged $\mathcal{N} = 8$ supergravity theory:

	helicity	$SU(4) \subset Sp(8)$	$d.o.f.$
$g_{\mu\nu}$	± 2	1	2
$\psi^I_{\mu\alpha}$	$\pm\frac{3}{2}$	$\square + \bar{\square}$	$4+4$
A^{IJ}_μ	± 1	$\ydiagram{1,1} + \ydiagram{1,1} + \ydiagram{2,1,1}$	$6+6+15$
χ^{IJK}	$\pm\frac{1}{2}$	$\square + \bar{\square} + \ydiagram{2,2,1} + \ydiagram{2,1}$	$4+4+20+20$
ϕ^{IJKL}	0	$1 + 1 + \ydiagram{2} + \overline{\ydiagram{2}} + \ydiagram{2,2}$	$2+10+10+20$

References

[1] J. Maldacena, "The large-N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231 hep-th/9711200.

[2] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," hep-th/9905111.

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. **B428** (1998) 105 hep-th/9802109.

[4] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 hep-th/9802150.

[5] H. J. Kim, L. J. Romans, P. Van Nieuwenhuizen, "Mass Spectrum of Chiral Ten-Dimensional $N = 2$ Supergravity on S^5," Phys.Rev.**D32** (1985) 389.

[6] M. Gunaydin, N. Marcus, "The Spectrum of The S^5 Compactification of Chiral $N = 2, D = 10$ Supergravity and The Unitary Supermultiplets of $U(2, 2/4)$," Class.Quant.Grav.**2** (1985) L11.